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Integrable Top Equations associated with Projective Geometry over Z_2

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ABSTRACT

We give a series of integrable top equations associated with the projective geometry over Z_2 as a $(2^n - 1)$-dimensional generalisation of the 3D Euler top equations. The general solution of the $(2^n - 1)$D top is shown to be given by an integration over a Riemann surface with genus $(2^{n-1} - 1)^2$.

1 $(2^n - 1)$D top equations

Recently we discovered an apparently new integrable set of evolution equations in seven dimensions, which are an analogue of the well-known 3D Euler top [1][2]. The 7D top arises from the dimensional reduction of the 8D $Spin(7)$ invariant self-dual Yang-Mills (SDYM) equations in [3], just as the 3D top comes from the reduction of the 4D SDYM to differential equations depending only upon one variable. The integrability of the 3D top is ensured by the existence of the Lax formulation of the 4D SDYM [4], while there is no such first order structure behind the 8D SDYM [5]. Nevertheless the 7D top has been shown to have sufficient conserved quantities to permit full integrability [1] and its general solution is given by a non-hyperelliptic differential equation corresponding to a Riemann surface with genus 9 [2].

The derivation of the top equations from the SDYM shows their connection with the existence of the division algebras, the 3D system arising from the quaternionic algebra, the 7D one from the octonions, which seems to suggest that no further integrable top system in more than seven dimensions should exist. In this note, however, we demonstrate that a generalisation of our previous results to general $2^n - 1$ dimensions is possible and is associated rather with the n-dimensional projective space over Z_2.

We take the projective space $Z_2 P_{n-1}$ with homogeneous coordinates $(z_0, z_1, \cdots, z_{n-1})$, where z_i is either 0,1 and calculations are performed in arithmetic mod 2. The space $Z_2 P_{n-1}$ consists of a finite number of points e_i $(i = 1, \cdots, 2^n - 1)$ with the multiplication operation $e_i e_j$ defined by the sum of their associated coordinates.

For the 3D $(n = 2)$ case, we have three points,

$$e_1 = (0, 1) \ , \qquad e_2 = (1, 0) \ , \qquad e_3 = (1, 1) \ , \tag{1}$$

with the multiplication rule,

$$e_i e_j = \varepsilon_{ijk}^2 \, e_k \ , \tag{2}$$

where ε_{ijk} is the structure constant of the $su(2)$ (quaternion) algebra. Using this structure constant, we obtain the 3D Euler top equations with variables $(\omega_1(t), \omega_2(t), \omega_3(t))$,

$$\frac{d}{dt}\omega_i = \frac{1}{2}\varepsilon_{ijk}^2 \, \omega_j \, \omega_k \ . \tag{3}$$

In the 7D $(n = 3)$ case, we have seven points,

$$\begin{aligned} & e_1 = (0, 0, 1) \ , \ e_2 = (0, 1, 0) \ , \ e_3 = (1, 0, 0) \ , \ e_4 = (1, 1, 1) \ , \\ & e_5 = (1, 1, 0) \ , \ e_6 = (1, 0, 1) \ , \ e_7 = (0, 1, 1) \ , \end{aligned} \tag{4}$$

with the relation

$$e_i e_j = c_{ijk}^2 \, e_k \tag{5}$$

where c_{ijk} is equal to a realization of the totally anti-symmetric structure constant appearing in the Cayley (octonion) algebra,

$$c_{127} = c_{631} = c_{541} = c_{532} = c_{246} = c_{734} = c_{567} = 1 \ . \qquad \text{(others zero)} \tag{6}$$

The relation (5) can be read off from the diagram in Fig.1, the seven-point plane with 7 points and 7 lines; 3 points lie on each line and 3 lines pass through each point. Replacing ε_{ijk} in (3) by the constant c_{ijk}, we obtain the set of seven equations for a 7D top [1][2],

$$\begin{aligned}
&\frac{d}{dt}\omega_1 = \omega_2\omega_7 + \omega_6\omega_3 + \omega_5\omega_4 \ , \quad &&\frac{d}{dt}\omega_2 = \omega_7\omega_1 + \omega_5\omega_3 + \omega_4\omega_6 \ , \\
&\frac{d}{dt}\omega_3 = \omega_1\omega_6 + \omega_2\omega_5 + \omega_4\omega_7 \ , \quad &&\frac{d}{dt}\omega_4 = \omega_1\omega_5 + \omega_6\omega_2 + \omega_7\omega_3 \ , \\
&\frac{d}{dt}\omega_5 = \omega_4\omega_1 + \omega_3\omega_2 + \omega_6\omega_7 \ , \quad &&\frac{d}{dt}\omega_6 = \omega_3\omega_1 + \omega_2\omega_4 + \omega_7\omega_5 \ , \\
&\frac{d}{dt}\omega_7 = \omega_1\omega_2 + \omega_3\omega_4 + \omega_5\omega_6 \ .
\end{aligned} \tag{7}$$

In a similar fashion to the above 3D and 7D cases, we can obtain $2^n - 1$ equations for a (2^n-1)D top. The structure of the higher-dimensional tops can be understood from the (2^n-1)-point hyperplane diagram, which is an extension of the seven-point plane and consists of $2^n - 1$ points and $2^n - 1$ $(2^{n-1}-1)$-point hyperplanes, where $2^{n-1}-1$ points lie on each $(2^{n-1}-1)$-point plane and $2^{n-1}-1$ $(2^{n-1}-1)$-point hyperplanes pass through each point.

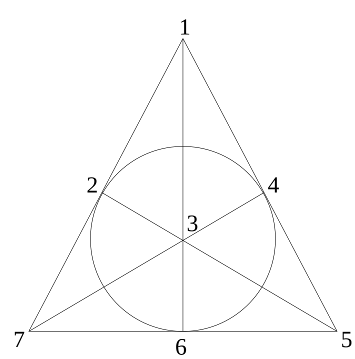

Figure 1: 7 point plane.

For example, in the 15D ($n = 4$) case with an appropriate labelling of 15 points in Z_2P_3, we have a 15-point tetrahedral space containing the following 15 7-point planes assigned by 7 points in them,

$$\begin{array}{lll}
(1,2,3,4,5,6,7)\ , & (1,2,8,11,10,9,7)\ , & (1,3,8,13,12,9,6)\ , \\
(2,3,8,14,12,10,5)\ , & (1,2,13,14,15,12,7)\ , & (1,3,14,11,10,15,6)\ , \\
(1,4,8,14,15,9,5)\ , & (1,4,13,11,10,12,5)\ , & (2,3,11,13,15,9,5)\ , \\
(2,4,8,13,15,10,6)\ , & (2,4,11,14,12,9,6)\ , & (3,4,8,11,15,12,7)\ , \\
(3,4,9,10,14,13,7)\ , & (5,6,8,11,13,14,7)\ , & (5,6,9,10,12,15,7)\ ,
\end{array} \tag{8}$$

where the p-th element in each of the above 15 brackets is placed on the point p in Fig.1. The form of the 15D top equations derived from the 15 point-hyperplane is

$$\begin{aligned}
\dot\omega_1 &= \omega_2\omega_7 + \omega_3\omega_6 + \omega_5\omega_4 + \omega_8\omega_9 + \omega_{10}\omega_{11} + \omega_{12}\omega_{13} + \omega_{14}\omega_{15}\ , \\
\dot\omega_2 &= \omega_1\omega_7 + \omega_3\omega_5 + \omega_4\omega_6 + \omega_8\omega_{10} + \omega_{11}\omega_9 + \omega_{12}\omega_{14} + \omega_{15}\omega_{13}\ , \\
\dot\omega_3 &= \omega_1\omega_6 + \omega_2\omega_5 + \omega_7\omega_4 + \omega_8\omega_{12} + \omega_9\omega_{13} + \omega_{10}\omega_{14} + \omega_{11}\omega_{15}\ , \\
\dot\omega_4 &= \omega_5\omega_1 + \omega_2\omega_6 + \omega_7\omega_3 + \omega_8\omega_{15} + \omega_9\omega_{14} + \omega_{10}\omega_{13} + \omega_{11}\omega_{12}\ , \\
\dot\omega_5 &= \omega_1\omega_4 + \omega_2\omega_3 + \omega_7\omega_6 + \omega_8\omega_{14} + \omega_9\omega_{15} + \omega_{10}\omega_{12} + \omega_{11}\omega_{13}\ , \\
\dot\omega_6 &= \omega_1\omega_3 + \omega_2\omega_4 + \omega_7\omega_5 + \omega_8\omega_{13} + \omega_9\omega_{12} + \omega_{10}\omega_{15} + \omega_{11}\omega_{14}\ , \\
\dot\omega_7 &= \omega_1\omega_2 + \omega_3\omega_4 + \omega_6\omega_5 + \omega_8\omega_{11} + \omega_9\omega_{10} + \omega_{12}\omega_{15} + \omega_{13}\omega_{14}\ , \\
\dot\omega_8 &= \omega_1\omega_9 + \omega_2\omega_{10} + \omega_3\omega_{12} + \omega_4\omega_{15} + \omega_5\omega_{14} + \omega_6\omega_{13} + \omega_7\omega_{11}\ , \\
\dot\omega_9 &= \omega_1\omega_8 + \omega_2\omega_{11} + \omega_3\omega_{13} + \omega_4\omega_{14} + \omega_5\omega_{15} + \omega_6\omega_{12} + \omega_7\omega_{10}\ , \\
\dot\omega_{10} &= \omega_1\omega_{11} + \omega_2\omega_8 + \omega_3\omega_{14} + \omega_4\omega_{13} + \omega_5\omega_{12} + \omega_6\omega_{15} + \omega_7\omega_9\ , \\
\dot\omega_{11} &= \omega_1\omega_{10} + \omega_2\omega_9 + \omega_3\omega_{15} + \omega_4\omega_{12} + \omega_5\omega_{13} + \omega_6\omega_{14} + \omega_7\omega_8\ , \\
\dot\omega_{12} &= \omega_1\omega_{13} + \omega_2\omega_{14} + \omega_3\omega_8 + \omega_4\omega_{11} + \omega_5\omega_{10} + \omega_6\omega_9 + \omega_7\omega_{15}\ , \\
\dot\omega_{13} &= \omega_1\omega_{12} + \omega_2\omega_{15} + \omega_3\omega_9 + \omega_4\omega_{10} + \omega_5\omega_{11} + \omega_6\omega_8 + \omega_7\omega_{14}\ , \\
\dot\omega_{14} &= \omega_1\omega_{15} + \omega_2\omega_{12} + \omega_3\omega_{10} + \omega_4\omega_9 + \omega_5\omega_8 + \omega_6\omega_{11} + \omega_7\omega_{13}\ , \\
\dot\omega_{15} &= \omega_1\omega_{14} + \omega_2\omega_{13} + \omega_3\omega_{11} + \omega_4\omega_8 + \omega_5\omega_9 + \omega_6\omega_{10} + \omega_7\omega_{12}\ .
\end{aligned} \tag{9}$$

2 General solution of the $(2^n - 1)$D top

2.1 Integrability

To show the integrability of the (2^n-1)D top and its general solution, it is convenient to work with a set of 2^n-1 variables a_i, instead of the ω_i's. The rule to define the a_i's is to pick up 2^n-1 sets of 2^{n-1} ω_i's which do not lie on a $(2^{n-1}-1)$-point subplane in the (2^n-1)-point hyperplane and to assign a_i to the sum of all 2^{n-1} ω_i's in each of the 2^n-1 sets. For example, in the 3D case,

$$a_1=\omega_2+\omega_3\ ,\qquad a_2=\omega_3+\omega_1\ ,\qquad a_3=\omega_1+\omega_2\ , \tag{10}$$

and in the 7D case,

$$\begin{aligned}
&a_1=\omega_3+\omega_4+\omega_5+\omega_6\ , &&a_2=\omega_1+\omega_2+\omega_5+\omega_6\ ,\\
&a_3=\omega_1+\omega_3+\omega_5+\omega_7\ , &&a_4=\omega_2+\omega_4+\omega_5+\omega_7\ ,\\
&a_5=\omega_2+\omega_3+\omega_6+\omega_7\ , &&a_6=\omega_1+\omega_4+\omega_6+\omega_7\ ,\\
&a_7=\omega_1+\omega_2+\omega_3+\omega_4\ .
\end{aligned} \tag{11}$$

Similarly, 15 variables a_i in the 15D top can be easily read off from the 15-point hyperplane defined in (8).

Using the variables a_i, the (2^n-1)D top equations are re-expressed as

$$\dot a_i=a_i\,(S-a_i)\ ,\qquad S=\frac{1}{2^{n-1}}\sum_{j=1}^{2^n-1}a_j\ , \tag{12}$$

and the equations of motion for the difference of the a_i's are

$$(a_i\dot{-}a_k)=(a_i-a_k)\,(S-a_i-a_k)\ . \tag{13}$$

We introduce the quantity W with the constants ρ_i and χ_{ij},

$$W=\sum_i\rho_i\ln a_i+\sum_{i<j}\chi_{ij}\ln\,(a_i-a_j)\ . \tag{14}$$

The condition $\dot W=0$ leads us to $(2^n-1)(2^{n-1}-1)$ conserved quantities N_{ij},

$$N_{ij}=T(a_i-a_j)/a_ia_j\ ,\quad T=(\prod_{k=1}^{2^n-1}a_k)^{\frac{1}{2^{n-1}-1}}\ . \tag{15}$$

Although the N_{ij} are not independent, they are sufficient to construct a basis of 2^n-2 independent conserved quantities, thus guaranteeing the integrability of the (2^n-1)D

top. Specifically, all the N_{ij} can be expressed in terms of N_{1j} $(j = 2, \cdots, 2^n - 1)$ through the relation

$$N_{ij} = N_{1j} - N_{1i} \ , \tag{16}$$

which means that any conserved quantities in the system can be constructed from these $2^n - 2$ quantities N_{1j}. In particular it is possible to define $2^n - 1$ polynomial conserved quantities γ_i from N_{ij} as

$$\gamma_i = N_{j_1k_1}N_{j_2k_2} \cdots N_{j_{2^{n-1}-1}k_{2^{n-1}-1}} = a_i(a_{j_1} - a_{k_1})(a_{j_2} - a_{k_2}) \cdots (a_{j_{2^{n-1}-1}} - a_{k_{2^{n-1}-1}}) \ , \tag{17}$$

where (j_p, k_p), $(j_p < k_p,\ p = 1, \cdots, 2^{n-1} - 1)$ lie on the respective $2^{n-1} - 1$ lines through the point i. The polynomials γ_i are of order 2^{n-1}. There is, of course, one functional relationship connecting these $2^n - 1$ expressions.

Summing over the index i of N_{ij} in (15), we see that all a_j are expressed in terms of two variables T and U, with the constants $M_j = \sum_{i=1}^{2^n-1} N_{ij}/(2^n - 1)$,

$$a_j^{-1} = M_j T^{-1} + U \ , \qquad U = \frac{1}{2^n - 1} \sum_{i=1}^{2^n-1} a_i^{-1} \ . \tag{18}$$

Note that the variables T and U are symmetric under any permutation of a_i's. Substituting the expression of a_i's into the definition of T in (15), we have the following relation between T and U,

$$T^{2^{n-1}} = \prod_{j=1}^{2^n-1} (T\,U + M_j) \ . \tag{19}$$

From (18) and (19), we see that all variables are expressible in terms of one variable, which demonstrates that the system of the $(2^n - 1)$D top is integrable. The explicit expression for the quadrature whose evaluation solves the top is given in the next subsection.

2.2 General solution

The time derivatives of T and U are derived from the equations of motion (12),

$$\dot{T} = T\,S \ , \qquad \dot{U} = -U\,S + 1 \ . \tag{20}$$

We introduce a variable $R(t) = T(t)\,U(t)$, whose time derivative is given as

$$\dot{R} = T \ . \tag{21}$$

Substituting $R = T\,U$ into (19) and (18), we have

$$T = (\prod_{j=1}^{2^n-1} (R + M_j))^{\frac{1}{2^{n-1}}} \ , \tag{22}$$

and

$$a_j = \frac{T}{(R+M_j)} = \frac{(\prod_{k=1}^{2^n-1}(R+M_k))^{\frac{1}{2^{n-1}}}}{(R+M_j)} \ . \tag{23}$$

Using (21) and (22), we obtain a first-order equation for $R(t)$,

$$\dot{R} = (\prod_{j=1}^{2^n-1}(R+M_j))^{\frac{1}{2^{n-1}}} \ , \tag{24}$$

which is non-hyperelliptic except for the 3D ($n=2$) case. The integral associated with this equation can be shown to correspond to a Riemann surface with genus $g = (2^{n-1}-1)^2$; the order $1/2^{n-1}$ in the RHS of (24) means that we need 2^{n-1} complex surfaces, each of which has 2^{n-1} cuts since the order of R is $2^n - 1$ inside the bracket of the RHS.

3 Further generalisations

It would be natural to expect that the examples of this note could be further generalised to the discussion of evolution equations for $\frac{k^n-1}{k-1}$ variables, corresponding to tops based upon the space $Z_k P_{n-1}$. Despite many efforts, we have as yet been unable to demonstrate a set of integrable equations for integral $k > 2$ except for the case where $n = 2$ and there are $k+1$ points lying on a line. Then one possibility for a set of integrable evolution equations is [6]

$$\frac{d}{dt}\omega_i = \prod_{j\neq i}\omega_j \ . \qquad (i = 1, \cdots, k+1) \tag{25}$$

These equations are reduced to a hyperelliptic differential equation for a $g = k-1$ Riemann surface. It would be surprising if there is no elegant integrable generalisation of these or similar evolution equations.

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